Operator-Theoretic De Bruijn Functional and Robust Safety Control Framework

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Abstract

This paper presents an operator-theoretic formulation of the De Bruijn functional as a unified quadratic metric on $L^2(\mathbb{R})$. The framework establishes equivalence between time-domain localization, frequency-domain representation, and positive self-adjoint operator structure. The functional is embedded into a stochastic control framework with belief-state dynamics, coherent risk measures, stochastic barrier functions, and robust dynamic programming. The resulting formulation provides a unified mathematical structure for safety assessment in safety-critical systems through localized energy, diagnostic residuals, and risk-weighted deviation metrics.

1 Introduction

Safety-critical systems require mathematically consistent measures that unify signal energy, uncertainty, and control risk. This work constructs a De Bruijn-type functional as a quadratic operator on a Hilbert space and embeds it into a stochastic control framework with robust decision-making under uncertainty.

2 Mathematical Kernel

Let $L^2(\mathbb{R})$ be a Hilbert space with Fourier transform

$$(Ff)(\xi) = \widehat{f}(\xi) := \int_{\mathbb{R}} f(x) e^{-2\pi i \xi x}\, dx.$$

Let $Z \in L^2(\mathbb{R})$, with center $T \in \mathbb{R}$, scale $H > 0$, and window

$$w_H(x) = \operatorname{sech}(x/H).$$

Define the windowed signal

$$f_{T,H}(x) = Z(T + x)\, w_H(x).$$

3 De Bruijn Functional

The functional is defined as

$$\Lambda(T, H) = \frac{1}{2H} \|f_{T,H}\|_{L^2}^2.$$

Using Plancherel's theorem,

$$\Lambda(T,H) = \frac{1}{2H}\int_{\mathbb{R}} |Z(T+x)|^2 \,\mathrm{sech}^2(x/H)\,dx.$$

This represents a localized energy functional weighted by a scale-dependent kernel.

4 Operator Formulation

Define translation and multiplication operators:

$$(\tau_T f)(x) = f(x-T), \qquad (M_{w_H} f)(x) = w_H(x) f(x).$$

Then

$$f_{T,H} = M_{w_H}\tau_T Z.$$

Define the positive self-adjoint operator

$$A_{T,H} = \tau_T^{-1} F^{-1} M_{|\widehat{w_H}|^2} F \tau_T.$$

Thus

$$\Lambda(T,H) = \langle Z, A_{T,H} Z\rangle_{L^2}.$$

5 Kernel Representation

The operator admits a convolution kernel representation:

$$(A_{T,H}Z)(t) = \int_{\mathbb{R}} K_H(t-s-T)\, Z(s)\, ds,$$

where $K_H \geq 0$ is the positive-definite kernel with $\widehat{K_H}(\xi) = |\widehat{w_H}(\xi)|^2$.
Therefore

$$\Lambda(T,H) = \int_{\mathbb{R}} |Z(t)|^2 \, K_H(t-T)\, dt.$$

6 Structural Properties

6.1 Positivity

$$\Lambda(T,H) \geq 0.$$

6.2 Definiteness

$$\Lambda(T,H) = 0 \quad \Longleftrightarrow \quad Z = 0 \text{ a.e. on } \mathrm{supp}(K_H).$$

6.3 Unitary Invariance

The functional is invariant under Fourier transform and translation due to unitary conjugation.

7 Stochastic Control Framework

Let (X_t, Y_t, U_t) be a controlled stochastic system. The belief state is

$$b_t \in \mathcal{B} := \Delta(X),$$

with update

$$b_{t+1} = T(b_t, U_t, Y_{t+1}).$$

The induced kernel is

$$T_\pi(b' \mid b) = P(b_{t+1} = b' \mid b_t = b,\, U_t = \pi(b)).$$

8 Cost and Risk Functional

The stage cost is

$$L(b,u) = \mathbb{E}_b[c_{\mathrm{op}}(X,u)] + \lambda_R \,\mathrm{CVaR}_\alpha(Z_b(u)) + \lambda_G \,\mathbb{E}_b[g(X,u)],$$

with Conditional Value-at-Risk

$$\mathrm{CVaR}_\alpha(Z) = \min_{\eta\in\mathbb{R}} \left(\eta + \frac{1}{1-\alpha}\mathbb{E}[(Z-\eta)_+]\right).$$

9 Safety Constraints

Define the safe set

$$\mathcal{B}_{\mathrm{safe}} = \{b : h(b) \geq 0\}.$$

Stochastic barrier condition:

$$\mathbb{E}[h(b_{t+1}) \mid b_t = b] \geq (1-\alpha_h)h(b),$$

which implies probabilistic invariance

$$P(b_t \in \mathcal{B}_{\mathrm{safe}},\ \forall t) \geq 1-\delta.$$

10 Robust Bellman Operator

$$(B_{\mathrm{rob}}V)(b) = \min_u \max_{T\in\mathcal{P}} \left[L(b,u) + \gamma \int V(b')\, T(db' \mid b,u)\right].$$

Contraction property:

$$\|B_{\mathrm{rob}}V - B_{\mathrm{rob}}W\|_\infty \leq \gamma \|V-W\|_\infty.$$

Thus a unique fixed point exists:

$$V^* = B_{\mathrm{rob}}V^*.$$

11 Optimal Control Policy

$$\pi^*(b) \in \arg\min_{u} \max_{T \in \mathcal{P}} \left[L(b,u) + \gamma \int V^*(b')\, T(db' \mid b,u) \right].$$

Guarantees:

$$P(b_t \in \mathcal{B}_{\text{safe}}, \forall t) \geq 1-\delta, \qquad \mathbb{E}\left[\sum_{t=0}^{\infty} \gamma^t L(b_t, \pi^*(b_t)) \right] \leq V^*(b_0).$$

12 Interpretation of the De Bruijn Functional

The functional $\Lambda(T, H)$ represents a localized quadratic observable with three interpretations:

- control error energy,
- diagnostic residual energy,
- risk-weighted deviation measure.

It acts as a positive self-adjoint operator linking:

- harmonic analysis (Fourier structure),
- stochastic control (belief MDPs),
- risk-sensitive decision systems (CVaR and safety constraints).

13 Conclusion

The De Bruijn functional provides a unified operator-theoretic structure that integrates localized energy measurement with stochastic control and robust safety frameworks. Its quadratic operator form ensures mathematical consistency, while its embedding in belief-space control theory enables applications in safety-critical decision systems.

References

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NOTES about References:

1. The foundational, citable work is older (1990–2020). Most of the mathematically rigorous material comes from:

harmonic analysis (1950–2005)
operator theory (1970–2015)
RKHS (1990–2016)
control theory (1990–2020)
risk theory (2000–2015)
functional safety (IEC 61508/61511, 2010–2016)

These areas simply do not publish new "core theory" every year.

2. 2025–2026 papers exist — but they are not foundational. Recent papers (2023–2026) tend to be:

applications of CBFs
reinforcement learning safety
data-driven diagnostics
safe MPC
risk-sensitive RL
digital twins for SIL verification
AI-based hazard prediction

These are extensions, not the mathematical backbone of operator-theoretic framework. "Our operator-theoretic functional aligns with modern 2024–2026 safety research."

"Our risk formulation is consistent with recent CVaR and SIL literature."

"Our diagnostic interpretation matches 2025 digital-twin fault detection trends."

"Our safety-control embedding is consistent with 2024–2026 CBF and RL safety work."

14 Mathematical Kernel

Let $(L^2(\mathbb{R}))$ be a Hilbert space with Fourier transform

$$\mathcal{F}f(\xi) = \hat{f}(\xi) := \int_{\mathbb{R}} f(x)e^{-2\pi i\xi x}\,dx.$$

Let $Z \in L^2(\mathbb{R})$, center $T \in \mathbb{R}$, scale $H > 0$, and window

$$w_H(x) := \operatorname{sech}\left(\frac{x}{H}\right).$$

Define the windowed signal

$$f_{T,H}(x) := Z(T + x)\,w_H(x).$$

15 De Bruijn Functional

Define

$$\Lambda(T, H) := \frac{1}{2H}\|f_{T,H}\|_{L^2}^2 = \frac{1}{2H}\|\hat{f}_{T,H}\|_{L^2}^2.$$

By Plancherel's theorem,

$$\Lambda(T, H) = \frac{1}{2H}\int_{\mathbb{R}} |Z(T + x)|^2 \operatorname{sech}^2\left(\frac{x}{H}\right)\,dx.$$

16 Operator Form

Define translation and multiplication operators

$$(\tau_T f)(x) = f(x - T), \quad (M_{w_H} f)(x) = w_H(x)f(x).$$

Then

$$f_{T,H} = M_{w_H}\tau_T Z.$$

Define the positive self-adjoint operator

$$A_{T,H} := \tau_T^{-1}\mathcal{F}^{-1}M_{|\hat{w}_H|^2}\mathcal{F}\tau_T.$$

Then the functional admits the quadratic form

$$\Lambda(T, H) = \langle Z, A_{T,H}Z\rangle_{L^2}.$$

17 Kernel Representation

Since $A_{T,H}$ is a Fourier multiplier conjugated by translation, it admits a convolution kernel K_H such that

$$(A_{T,H}Z)(t) = \int_{\mathbb{R}} K_H(t - s - T)Z(s)\,ds.$$

Hence

$$\Lambda(T, H) = \int_{\mathbb{R}} |Z(t)|^2 K_H(t - T)\,dt, \quad K_H \geq 0.$$

18 Structural Properties

(i) Positivity:

$$\Lambda(T, H) \geq 0.$$

(ii) Definiteness:

$$\Lambda(T, H) = 0 \iff Z = 0 \text{ a.e. on support of } K_H.$$

(iii) Unitary invariance: Λ is invariant under Fourier transform and translation due to unitary conjugation.

19 Stochastic Control System

Let (X_t, Y_t, U_t) be a controlled stochastic system with belief state

$$b_t \in \mathcal{B} := \Delta(\mathcal{X}), \quad b_t(A) = \mathbb{P}(X_t \in A \mid Y_{0:t}, U_{0:t-1}).$$

Bayesian update:

$$b_{t+1} = \mathcal{T}(b_t, U_t, Y_{t+1}).$$

Induced kernel:

$$T_\pi(b'|b) = \mathbb{P}(b_{t+1} = b' \mid b_t = b, U_t = \pi(b)).$$

20 Cost and Risk

Stage cost:

$$L(b, u) = \mathbb{E}_b[c_{\text{op}}(X, u)] + \lambda_R \text{CVaR}_\alpha(Z_b(u)) + \lambda_G \mathbb{E}_b[g(X, u)].$$

CVaR:

$$\text{CVaR}_\alpha(Z) = \min_{\eta \in \mathbb{R}} \left(\eta + \frac{1}{1-\alpha} \mathbb{E}[(Z - \eta)^+] \right).$$

21 Safety Constraints

Safe set:

$$\mathcal{B}_{\text{safe}} = \{b : h(b) \geq 0\}.$$

Stochastic barrier condition:

$$\mathbb{E}[h(b_{t+1})|b_t = b] \geq (1 - \alpha_h) h(b).$$

This implies probabilistic invariance:

$$\mathbb{P}(b_t \in \mathcal{B}_{\text{safe}}, \forall t) \geq 1 - \delta.$$

22 Robust Bellman Operator

$$(\mathcal{B}_{\text{rob}} V)(b) = \min_u \max_{T \in \mathcal{P}} \left[L(b, u) + \gamma \int V(b')\, T(db'|b, u) \right].$$

Under boundedness and rectangularity:

$$\|\mathcal{B}_{\text{rob}} V - \mathcal{B}_{\text{rob}} W\|_\infty \leq \gamma \|V - W\|_\infty.$$

Thus a unique fixed point exists:

$$V^* = \mathcal{B}_{\text{rob}} V^*.$$

23 Final Coupled System

The optimal policy is

$$\pi^*(b) \in \arg\min_u \max_{T\in\mathcal{P}} \left[L(b,u) + \gamma \int V^*(b')T(db'|b,u) \right].$$

Safety and robustness guarantees:

$$\mathbb{P}(b_t \in \mathcal{B}_{\text{safe}}, \forall t) \geq 1-\delta, \quad \mathbb{E}\left[\sum_{t=0}^{\infty} \gamma^t L(b_t, \pi^*(b_t)) \right] \leq V^*(b_0).$$

24 Interpretation of $\Lambda(T, H)$

The functional $\Lambda(T, H)$ serves as a localized quadratic observable:

- energy deviation (control error),
- diagnostic residual energy,
- risk-weighted deviation measure.

It is a positive self-adjoint operator functional acting as a bridge between harmonic analysis, stochastic control, and risk-aware safety systems.